机械与运载工程学院

机械CAE基础

主讲：张见明

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提交作业的邮件名称＝学号＋姓名＋作业号

例如：s09022222扬帆第1次作业

有关本课程的几点说明

课时：32学时（8学时/周，共4周）
授课方式：大课讲解（32学时）

考试（100分满分）：1. 平时作业---20% .
　　　　　　　　　　2. 平时提问---10% .
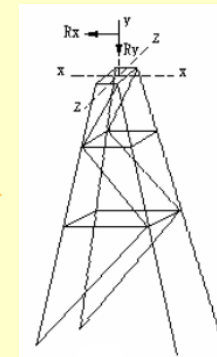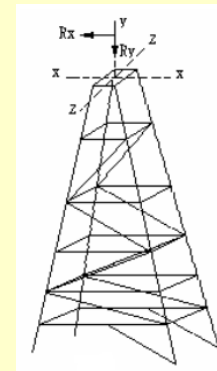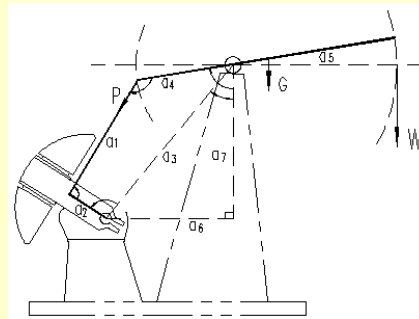　　　　　　　　　　3. 编写程序---20% .
　　　　　　　　　　4. 结束笔试---50% .

笔试方式：开卷
笔试时间：待定
程序考核时间：待定
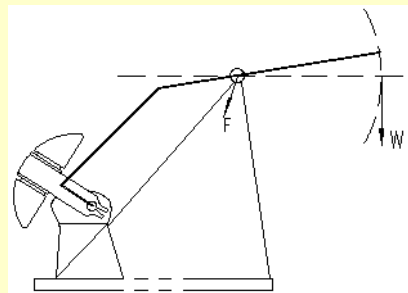
参考教材：
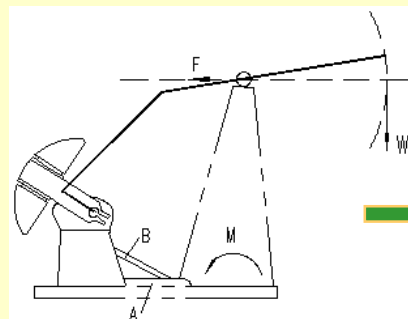
1、朱郁森、刘金枝主编， 数学物理方程，湖南大学出版社。

2、王勖成等， 有限元法基本原理和数值方法，清华大学出版社。

3、曾金平主编， 数值计算方法，湖南大学出版社。

4、李庆阳等， 数值分析，清华大学出版社。

Examples of industrial work

- Sampson post of pumping unit (1991-1992)

➤ Further improvement

Mass:	Mass:
2280kg	1421kg
Displacement:	Displacement:
4.4 mm	3.1 mm
Max. stress:	Max. stress:
132 MPa	58 MPa

Examples of industrial work (2)

- Workover rig mast (1993-1994)
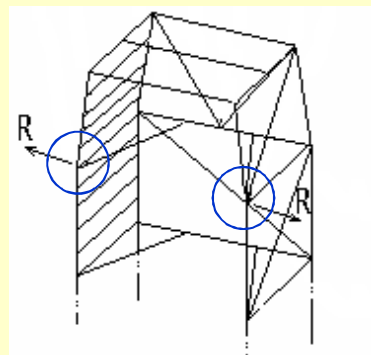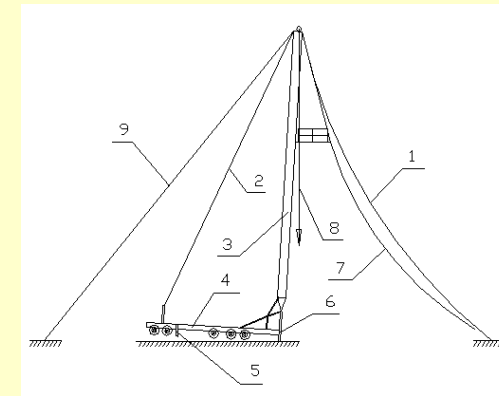
Max. stress: 325 MPa
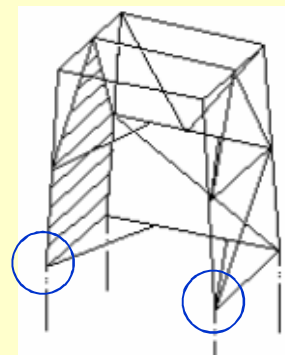
Max. stress: 180 MPa

Examples of industrial work (3)

➢ Further studies
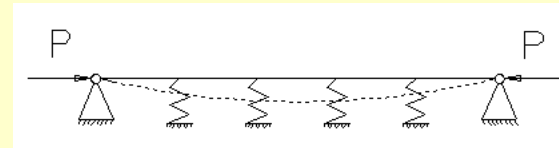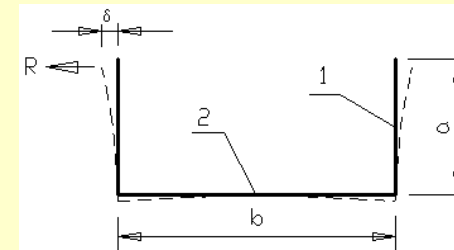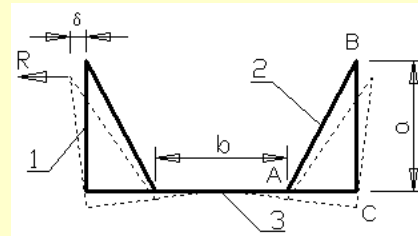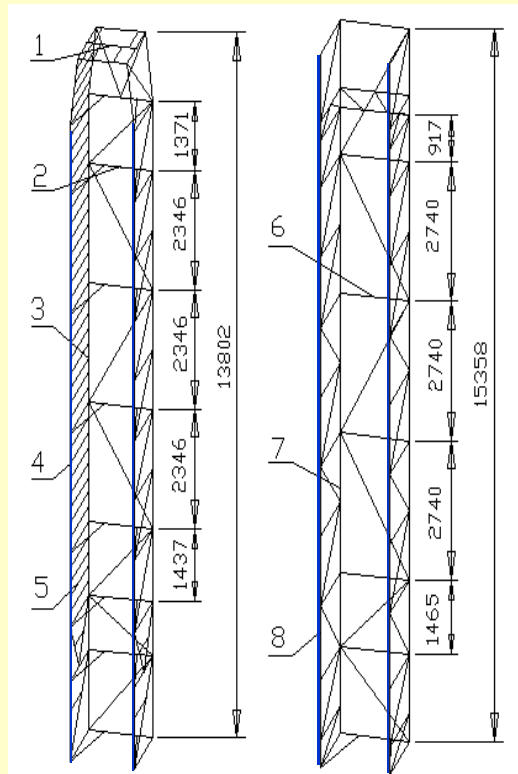
L140X140X12/16Mn ➡ I160×88×6/A3

a 半行式　　　b 波浪式　　　c 半行人字形　　　d 交叉人字形
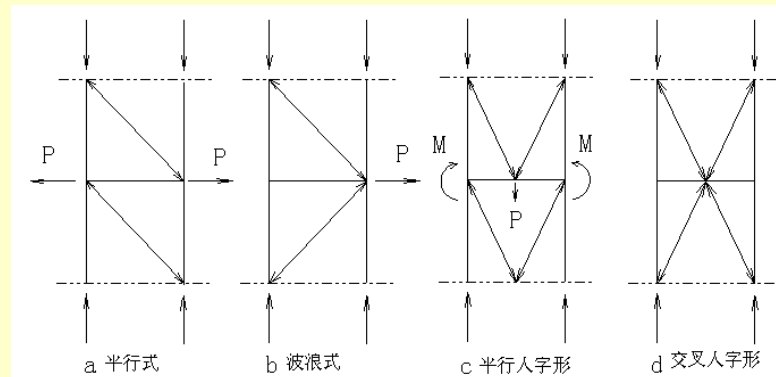
Examples of industrial work (4)

- FRP sucker rod End-fitting (1997-1998)
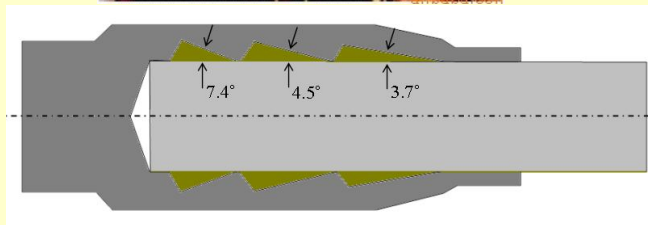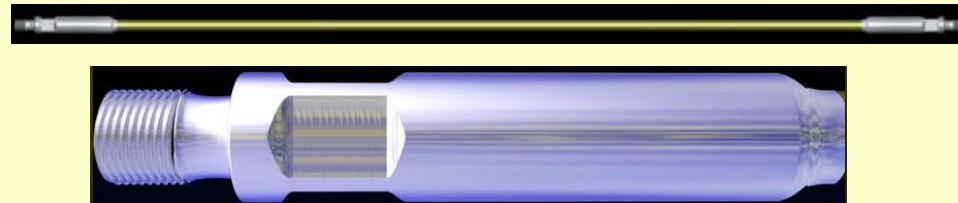
Fatigue cycles: 6.9 million
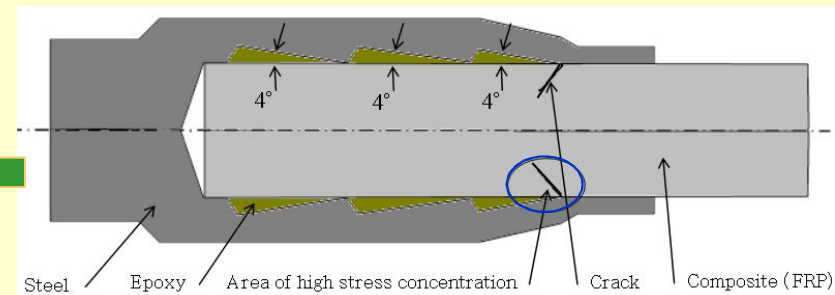
Fatigue cycles: 2.4 million

Steel   Epoxy   Area of high stress concentration   Crack   Composite (FRP)

- ➢ Idea for further improvement
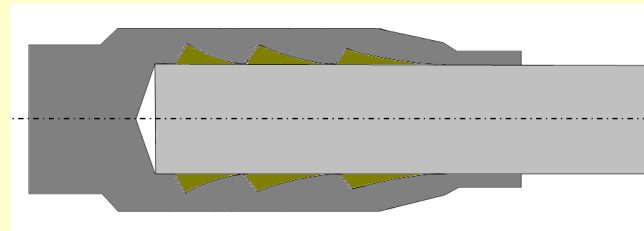
Examples of industrial work (5)

- 龙门导轨磨床(2011)
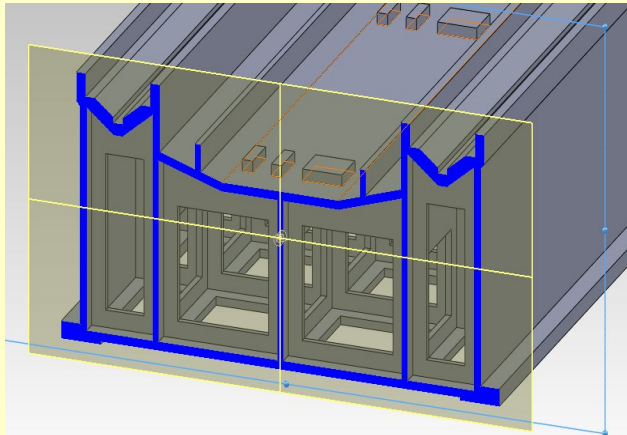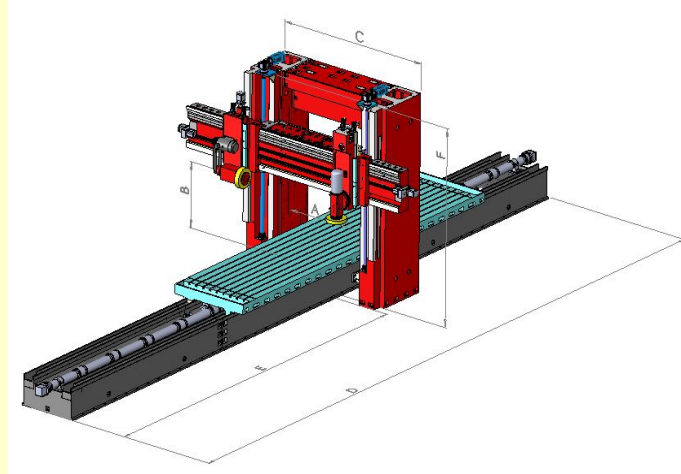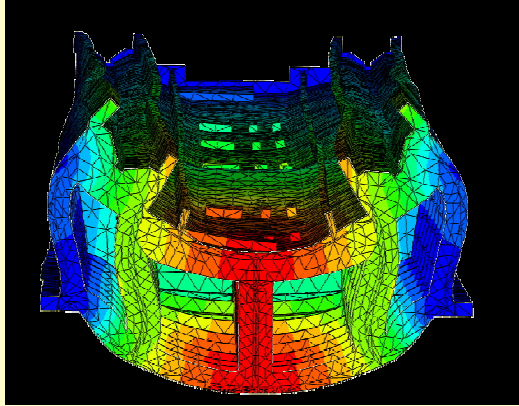
床身重量：39.598吨

导轨最大变形： 8.1e^{-2}mm
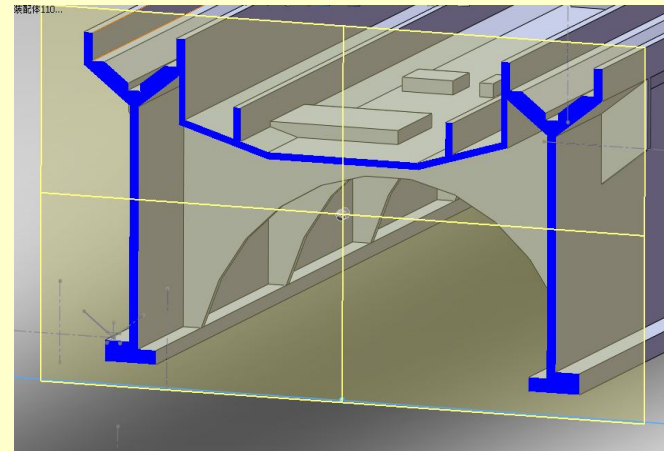
床身重量：25.274吨

导轨最大变形： 6.5e^{-2} mm

Examples of industrial work (6)

- Forming process of spiral welded pipes (1999)

形状尺寸设计 → CAD（AutoCAD, NX-UG, Pro/E等）

机械设计

物理性能设计 → CAE（Ansys, Nastran, Abacus等）

CAE ↔ 物理实验

CAE分析全过程

real system

physical model

physical theory: electromagnetism, continuum mechanics, nuclear physics, ...

assumptions: static, harmonic, transient analysis
2D, 3D, axisymmetric, shell, ...
type of material (elastic, plastic, ...)
kinematics (large deformations, ...)
load case

mathematical model

choice of coordinate system, units, independent variables

differential equations, integral equations
boundary and initial conditions
constraints

numerical model for FEM, BEM, FDM, ...

element type: triangular, no. of nodes, integration order, ...
mesh density: coarse, locally refined, ...
solution parameters: error bounds, tolerances,
time integration, iteration, ...

accuracy of solution
of mathematical model

refine num. model,
adaptivity

interpretation of results:
experience, common sense

improve phys. model,
modify assumptions

engineering solution

微分方程的建立

➤ **弦的振动**

弧段 $\overset{\frown}{MM'}$ 在 x 方向平衡：

$$T'\cos\alpha' - T\cos\alpha = 0$$

根据微小振动假设，即：$\alpha \approx 0, \alpha' \approx 0$

由　$\cos\alpha = 1 - \dfrac{\alpha^2}{2!} + \dfrac{\alpha^4}{4!} - \cdots$

略去二阶小量得到　$\cos\alpha \approx 1, \cos\alpha' \approx 1$

于是　　$T = T'$

弧段 $\widehat{MM'}$ 在 **u** 方向的运动方程：

$$-T\sin\alpha + T'\sin\alpha' - \rho g ds \approx \rho ds \frac{\partial^2 u(x,t)}{\partial t^2}$$

其中

$$\sin\alpha = \frac{tg\alpha}{\sqrt{1 + tg^2\alpha}} \approx tg\alpha = \frac{\partial u(x,t)}{\partial x}$$

$$\sin\alpha' \approx tg\alpha' = \frac{\partial u(x+dx,t)}{\partial x}$$

$$ds = \sqrt{1 + \left[\frac{\partial u(x,t)}{\partial x}\right]^2} dx \approx dx$$
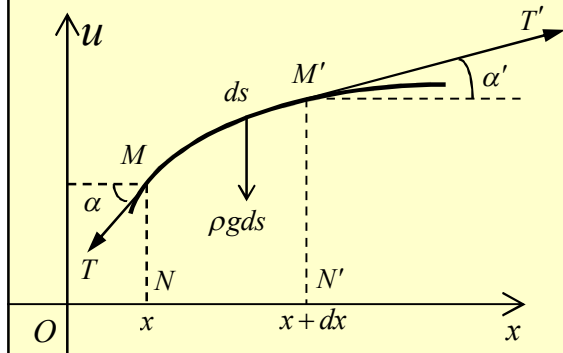
于是

$$T\left[\frac{\partial u(x+dx,t)}{\partial x} - \frac{\partial u(x,t)}{\partial x}\right] - \rho g dx \approx \rho \frac{\partial^2 u(x,t)}{\partial t^2} dx$$

$$T\left[\frac{\partial u(x+dx,t)}{\partial x}-\frac{\partial u(x,t)}{\partial x}\right]-\rho g dx \approx \rho \frac{\partial^2 u(x,t)}{\partial t^2}dx$$

由于

$$\frac{\partial u(x+dx,t)}{\partial x}-\frac{\partial u(x,t)}{\partial x} \approx \frac{\partial}{\partial x}\left[\frac{\partial u(x,t)}{\partial x}\right]dx=\frac{\partial^2 u(x,t)}{\partial x^2}dx$$

得到

$$\left[T\frac{\partial^2 u(x,t)}{\partial x^2}-\rho g\right]dx \approx \rho \frac{\partial^2 u(x,t)}{\partial t^2}dx$$

或者 $$\frac{T}{\rho}\frac{\partial^2 u(x,t)}{\partial x^2} \approx \frac{\partial^2 u(x,t)}{\partial t^2}+g$$

忽略g，最后得到 $$\frac{\partial^2 u}{\partial t^2}=a^2\frac{\partial^2 u}{\partial x^2} \qquad a^2=\frac{T}{\rho}$$
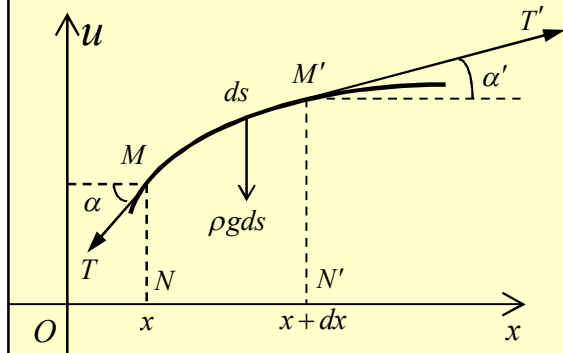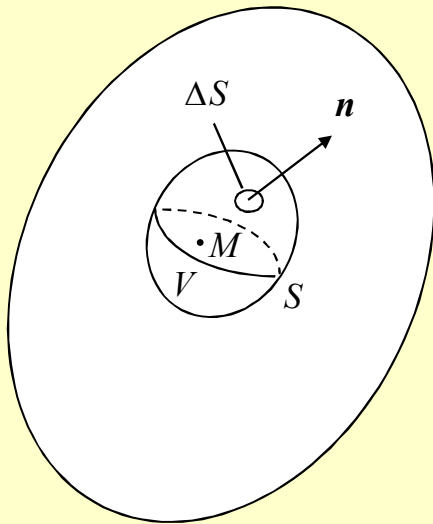
➤ 热传导方程　　　　对于任意体积 V 的表面 S 上的微元 $\triangle S$，根据 **Fourier** 热传导定理，在 dt 时间内，有：

$$dQ = -k\frac{\partial u}{\partial n}dSdt$$

$$= -k(\mathrm{grad}u)_n dSdt = -k\mathrm{grad}u \cdot d\boldsymbol{S}dt$$

则从 t_1 到 t_2 时刻，通过 $\boldsymbol{S}$ 流入区域 V 的全部热量为

$$Q_1 = \int_{t_1}^{t_2}\left[\iint_S k\mathrm{grad}u \cdot d\boldsymbol{S}\right]dt$$

根据奥-高公式

$$\iint_S k\mathrm{grad}u \cdot d\boldsymbol{S} = \iiint_V k\mathrm{div}\,\mathrm{grad}udV = \iiint_V k\nabla^2 udV$$

得到　　　　$$Q_1 = \int_{t_1}^{t_2}\left[\iiint_V k\nabla^2 udV\right]dt$$
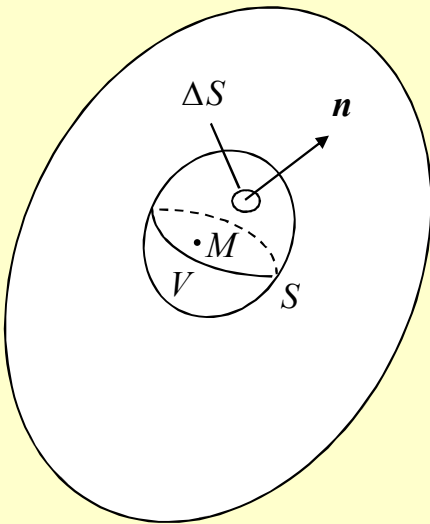
区域V内各点的温度从$u(x,y,z,t_1)$变化到$u(x,y,z,t_2)$所需要的热量为

$$\iiint\limits_{V} c\rho \left[u(x,y,z,t_2) - u(x,y,z,t_1) \right] dV$$

$$= \iiint\limits_{V} c\rho (\int_{t_1}^{t_2} \frac{\partial u}{\partial t} dt) dV = \int_{t_1}^{t_2} \left[\iiint\limits_{V} c\rho \frac{\partial u}{\partial t} dV \right] dt$$

由于能量守恒，有

$$\int_{t_1}^{t_2} \left[\iiint\limits_{V} k\nabla^2 u dV \right] dt = \int_{t_1}^{t_2} \left[\iiint\limits_{V} c\rho \frac{\partial u}{\partial t} dV \right] dt$$

由于时间间隔$[t_1, t_2]$及域V都是任意取的，上式中被积函数应恒等，即

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \qquad a^2 = \frac{k}{c\rho}$$

$$\frac{\partial u}{\partial t} = a^2 \nabla^2 u = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \qquad a^2 = \frac{k}{c\rho}$$
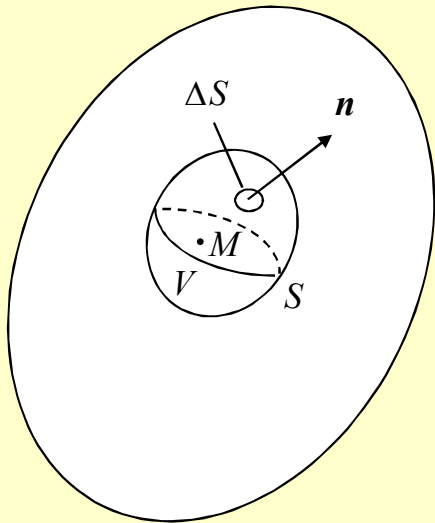
在上面的推导中，假定了k是常数。

若k随位置变化，即是坐标的函数$k(x,y,z)$，则该问题是非均质问题；

若k依赖温度，即是温度的函数$k(u)$，则问题变成非线性问题；

若考虑稳态温度场，温度不随时间变化，则得到拉普拉斯方程：

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

➢ 弹性力学方程
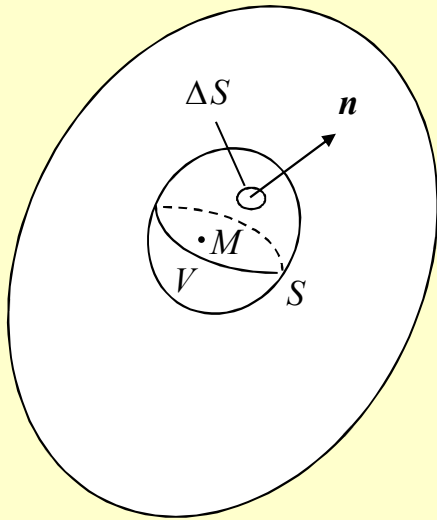
根据**Newton**第二定理：

$$F = ma$$

对于体积V有

$$\iint\limits_{S} t \cdot dS + \iiint\limits_{V} f dV = \iiint\limits_{V} \rho \ddot{u} dV$$

根据奥-高公式

$$\iint\limits_{S} t dS = \iint\limits_{S} n \Box \sigma dS = \iiint\limits_{V} \nabla \Box \sigma dV$$

考虑V的任意性，得到连续体的运动方程

$$\nabla \Box \sigma + f = \rho \ddot{u} \qquad \forall x \in V$$

对于静力问题，得到平衡方程

$$\nabla \Box \sigma + f = 0 \qquad \forall x \in V$$

$$\nabla \square \sigma + f = 0 \qquad \forall x \in V$$

引入几何方程

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad \text{或} \quad \varepsilon = \frac{1}{2}\left(\nabla u + u\nabla\right)$$

和本构方程

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + G\left(u_{i,j} + u_{j,i}\right) \quad \text{或} \quad \sigma = \lambda\left[\left(\varepsilon : I\right)I + 2G\varepsilon\right]$$

得到Lame-Navier方程

$$\left(\lambda + G\right)u_{j,ji} + Gu_{i,jj} + f_i = 0$$

或

$$\left(\lambda + G\right)\nabla\nabla\square u + G\nabla\square\nabla u + f = 0 \qquad \forall x \in V$$
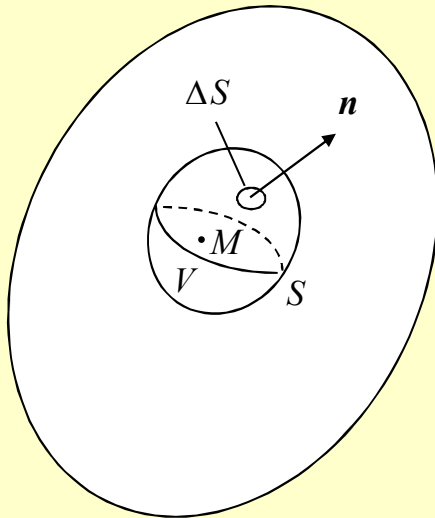
初始条件与边界条件

■ **初始条件**　说明某一具体物理现象的初始状态的条件称为初始条件。

➢ **弦的振动**

$$\begin{cases} u\big|_{t=0} = \varphi(x) \\ \dfrac{\partial u}{\partial t}\bigg|_{t=0} = \psi(x) \end{cases}$$

➢ **热传导问题**

$$u(M,t)\big|_{t=0} = \varphi(M)$$

➢ **弹性力学问题**

$$\begin{cases} \boldsymbol{u}(M,t)\big|_{t=0} = \boldsymbol{\varphi}(M) \\ \dfrac{\partial \boldsymbol{u}(M,t)}{\partial t}\bigg|_{t=0} = \boldsymbol{\psi}(M) \end{cases}$$

■ **边界条件**　说明某一具体物理现象的域边界上的约束情况的条件称为边界条件。

➢ **弦的振动**

■ **固定端**

$$u\big|_{x=a} = 0$$

■ **自由端**

$$T\frac{\partial u}{\partial x}\bigg|_{x=a} = 0 \quad \text{或} \quad \frac{\partial u}{\partial x}\bigg|_{x=a} = 0$$

■ **弹性支承端**

$$T\frac{\partial u}{\partial x}\bigg|_{x=a} = -k\,u\big|_{x=a} \quad \text{或} \quad \left(\frac{\partial u}{\partial x} + \sigma u\right)\bigg|_{x=a} = 0, \quad \sigma = k/T$$

➤ 热传导问题

■ **恒温边界**（Dirichlet边界条件）

$$u|_S = f$$

■ **绝热边界**（Neumann边界条件）

$$\left.\frac{\partial u}{\partial x}\right|_S = 0$$

■ **热交换边界**（Robin边界条件）

$$-k\left.\frac{\partial u}{\partial x}\right|_S = k_1\left.(-u_1)\right|_S \quad \text{或} \quad \left.\left(\frac{\partial u}{\partial x} + \sigma u\right)\right|_S = \sigma u_1, \quad \sigma = k_1/k$$

➢ 弹性力学问题

  ■ 位移边界条件

$$\left.\boldsymbol{u}\right|_{S_u} = \bar{\boldsymbol{u}}$$

  ■ 面力边界条件

$$\left.\boldsymbol{\sigma}\cdot\boldsymbol{n}\right|_{S_\sigma} = \bar{\boldsymbol{T}}$$

  或

$$\sigma_{11}n_1 + \sigma_{12}n_2 + \sigma_{13}n_3 = \bar{T}_1$$
$$\sigma_{21}n_1 + \sigma_{22}n_2 + \sigma_{23}n_3 = \bar{T}_2$$
$$\sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3 = \bar{T}_3$$

  ■ 接触边界条件

  略
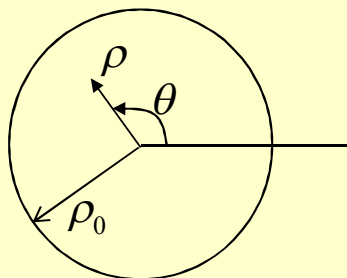
拉普拉斯方程 $\quad \nabla^2 u = 0$

边界条件 $\quad u\big|_{\rho=\rho_0} = f(\theta)$

在极坐标系下

$$\begin{cases} \nabla^2 u = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial u}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial\theta^2} = 0, \ \ \rho < \rho_0 \\ u(\rho_0,\theta) = f(\theta) \end{cases}$$

令 $\quad u(\rho,\theta) = R(\rho)\Phi(\theta)$

得到 $\quad R''\Phi + \dfrac{1}{\rho}R'\Phi + \dfrac{1}{\rho^2}R\Phi'' = 0$

即 $\quad \dfrac{\rho^2 R'' + \rho R'}{R} = -\dfrac{\Phi''}{\Phi}$

令　$\dfrac{\rho^2 R'' + \rho R'}{R} = -\dfrac{\Phi''}{\Phi} = \lambda$

得到

$$\Phi'' + \lambda\Phi = 0$$

$$\rho^2 R'' + \rho R' - \lambda R = 0$$

由于

$$|R(0)| < +\infty$$

$$\Phi(\theta + 2\pi) = \Phi(\theta)$$

得到

$$\begin{cases} \Phi'' + \lambda\Phi = 0 \\ \Phi(\theta + 2\pi) = \Phi(\theta) \end{cases} \qquad （1）$$

及

$$\begin{cases} \rho^2 R'' + \rho R' - \lambda R = 0 \\ |R(0)| < +\infty \end{cases} \qquad （2）$$

对于问题（1）

$$\begin{cases} \varPhi'' + \lambda \varPhi = 0 \\ \varPhi(\theta + 2\pi) = \varPhi(\theta) \end{cases}$$

当 $\lambda < 0$ 时，无非零解；

当 $\lambda = 0$ 时，解为 $\varPhi_0(\theta) = a_0'$ （常数）；

当 $\lambda = 0$ 时，取 $\lambda = \beta^2$，解为

$$\varPhi_\beta(\theta) = a_\beta' \cos \beta\theta + b_\beta' \sin \beta\theta$$

为使 $\varPhi(\theta)$ 以 2π 为周期，β 必须是整数 n，于是

$$\varPhi_n(\theta) = a_n' \cos n\theta + b_n' \sin n\theta$$

对于问题（**2**）

$$\begin{cases} \rho^2 R'' + \rho R' - \lambda R = 0 \\ \left| R(0) \right| < +\infty \end{cases}$$

当 $\lambda = 0$ 时，解为

$$R_0 = c_0 + d_0 \ln \rho$$

当 $\lambda = n^2$ 时，解为

$$R_n = c_n \rho^n + d_n \rho^{-n} \qquad (n = 1, 2, 3, \cdots)$$

为保证 $\left| R(0) \right| < +\infty$

$$R_n = c_n \rho^n \qquad (n = 1, 2, 3, \cdots)$$

结合问题（**1**）的解，有

$$u(\rho, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\theta + b_n \sin n\theta)$$

$$u(\rho,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho^n (a_n \cos n\theta + b_n \sin n\theta)$$

利用边界条件 $u\big|_{\rho=\rho_0} = f(\theta)$， 得

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \rho_0^n (a_n \cos n\theta + b_n \sin n\theta)$$

因此，$a_0, \rho_0^n a_n, \rho_0^n b_n$

就是 $f(\theta)$ 展开为 **Fourier** 级数时的系数，即有

$$\begin{cases} a_0 = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} f(\theta)d\theta \\[4mm] a_n = \dfrac{1}{\rho_0^n \pi} \displaystyle\int_0^{2\pi} f(\theta)\cos n\theta d\theta \\[4mm] b_n = \dfrac{1}{\rho_0^n \pi} \displaystyle\int_0^{2\pi} f(\theta)\sin n\theta d\theta \end{cases}$$

作业：

1. 有一均匀杆，只要杆中任一小段有纵向位移或速度，必导致邻段的压缩或伸长，这种伸缩传开去，就有纵波沿着杆传播。试推导杆的纵振动方程。

2. 混凝土内部储藏着热量，称为水化热，在它浇筑后逐渐释放，放热速度和它所储藏的水化热成正比。以 $Q(t)$ 表示它在单位体积中所储存的热量，Q_0 初始时刻所储存的热量，则 $dQ/dt = -\beta Q$，其中 β 为正常数。又假设混凝土的比热为 c、密度为 ρ，热传导系数为 k，求它在浇筑后温度 u 满足的方程。

有限差分法（FDM）

由微分学知道

$$u'(x) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - u(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{u(x) - u(x - \Delta x)}{\Delta x}$$

$$u''(x) = \lim_{\Delta x \to 0} \frac{u'(x + \Delta x) - u'(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left[\frac{u(x + \Delta x) - u(x)}{\Delta x} - \frac{u(x) - u(x - \Delta x)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \frac{u(x + \Delta x) - 2u(x) + u(x - \Delta x)}{(\Delta x)^2}$$

当 $|\Delta x|$ 很小时

$$u'(x) \approx \frac{u(x + \Delta x) - u(x)}{\Delta x} \quad \text{或} \quad u'(x) \approx \frac{u(x) - u(x - \Delta x)}{\Delta x}$$

$$u''(x) \approx \frac{u(x + \Delta x) - 2u(x) + u(x - \Delta x)}{(\Delta x)^2}$$

因此，拉普拉斯方程 $\quad \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

可以近似为

$$\frac{u(x + \Delta x, y) - 2u(x, y) + u(x - \Delta x, y)}{(\Delta x)^2}$$

$$+ \frac{u(x, y + \Delta y) - 2u(x, y) + u(x, y - \Delta y)}{(\Delta y)^2} = 0$$
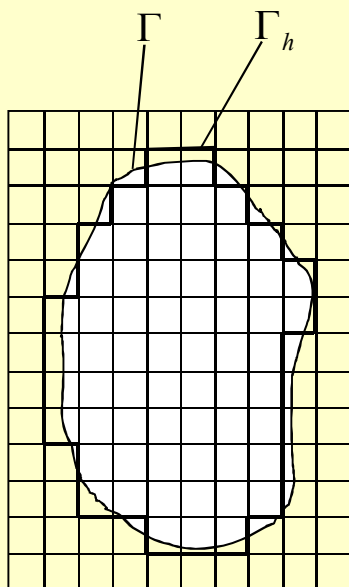
用平行于坐标轴的两族直线

$$x_i = x_0 + ih \qquad y_i = y_0 + ih$$

将求解域 Ω 划分为规则的网格，于是

$$\frac{\partial^2 u}{\partial x^2}\Big|_{(x_i, y_j)} \approx \frac{u(x_i + h, y_j) - 2u(x_i, y_j) + u(x_i - h, y_j)}{h^2}$$

$$\frac{\partial^2 u}{\partial y^2}\Big|_{(x_i, y_j)} \approx \frac{u(x_i, y_j + h) - 2u(x_i, y_j) + u(x_i, y_j - h)}{h^2}$$
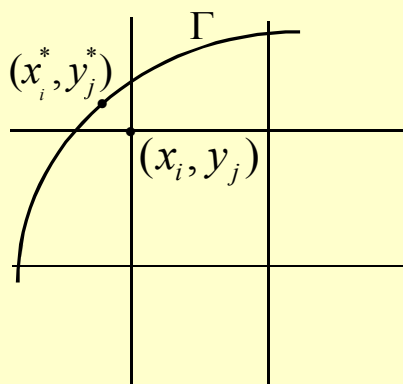
将边界 Γ 近似为 Γ_h ，规定

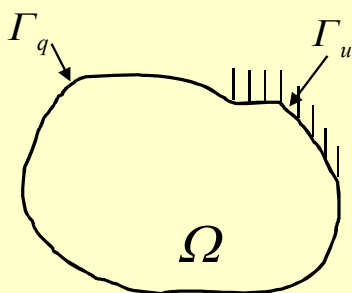$$u\big|_{(x_i, y_j) \in \Gamma_h} = f(x_i^*, y_j^*)$$

于是拉普拉斯方程变为

$$U_{i+1,j} + U_{i,j+1} + U_{i-1,j} + U_{i,j-1} - 4U_{i,j} = 0$$

其中 $U_{i,j}$ 表示解 $u(x, y)$ 在结点 (x_i, y_j) 处的近似值

有限元法（FEM）

二维稳态热传导问题

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad （在 \boldsymbol{\Omega} 内）$$

$$u - \bar{u} = 0 \qquad （在 \boldsymbol{\Gamma}_u 上）$$

$$k\frac{\partial u}{\partial n} - \bar{q} = 0 \qquad （在 \boldsymbol{\Gamma}_q 上）$$

$$\Gamma_u \bigcup \Gamma_q = \Gamma, \ \ \Gamma_u \bigcap \Gamma_q = \varnothing$$

等效弱积分形式

$$\int_{\Omega} v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dxdy + \int_{\Gamma_q} \bar{v}\left(k\frac{\partial u}{\partial n} - \bar{q} \right) d\Gamma = 0$$

其中 v 为任意函数

由于

$$\int_{\Omega} v \frac{\partial^2 u}{\partial x^2} dxdy = -\int_{\Omega} \frac{\partial v}{\partial x} \frac{\partial u}{\partial x} dxdy + \int_{\Omega} \frac{\partial}{\partial x}\left(v \frac{\partial u}{\partial x}\right) dxdy$$

$$\int_{\Omega} v \frac{\partial^2 u}{\partial y^2} dxdy = -\int_{\Omega} \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} dxdy + \int_{\Omega} \frac{\partial}{\partial y}\left(v \frac{\partial u}{\partial y}\right) dxdy$$
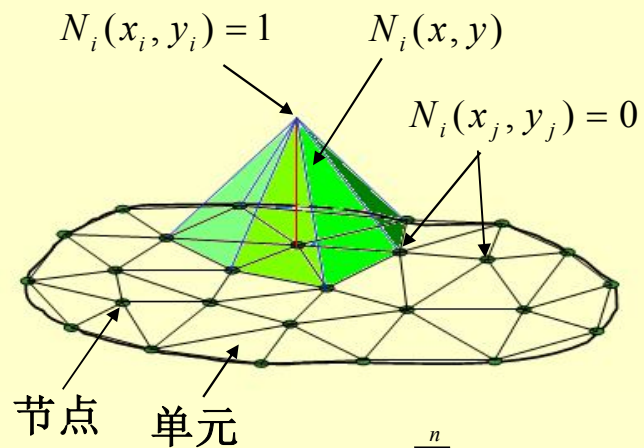
得到

$$\int_{\Omega} v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dxdy$$

$$= -\int_{\Omega} \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} \right) dxdy + \int_{\Omega} \frac{\partial}{\partial x}\left(v \frac{\partial u}{\partial x}\right) dxdy + \int_{\Omega} \frac{\partial}{\partial y}\left(v \frac{\partial u}{\partial y}\right) dxdy$$

$$= -\int_{\Omega} \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} \right) dxdy + \oint_{\Gamma} v \frac{\partial u}{\partial x} n_x + v \frac{\partial u}{\partial y} n_y d\Gamma$$

$$= -\int_{\Omega} \left(\frac{\partial v}{\partial x} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial u}{\partial y} \right) dxdy + \oint_{\Gamma} v \frac{\partial u}{\partial n} d\Gamma$$

于是

$$\int_{\Omega} v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dxdy + \int_{\Gamma_q} \overline{v} \left(\frac{\partial u}{\partial n} - \overline{q} \right) d\Gamma$$

$$= -\int_{\Omega} \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \right) dxdy + \oint_{\Gamma} v \frac{\partial u}{\partial n} d\Gamma + \int_{\Gamma_q} \overline{v} \left(\frac{\partial u}{\partial n} - \overline{q} \right) d\Gamma$$

$$= -\int_{\Omega} \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \right) dxdy + \int_{\Gamma_q} \left(v + \overline{v} \right) \frac{\partial u}{\partial n} d\Gamma + \int_{\Gamma_u} v \frac{\partial u}{\partial n} d\Gamma - \int_{\Gamma_q} \overline{v}\,\overline{q} d\Gamma$$

$$= 0$$

假定 $v = -\overline{v}$ ，且在 Γ_u 边界上 $v = 0$ ，得到

$$\int_{\Omega} \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \right) dxdy = \int_{\Gamma_q} v\,\overline{q} d\Gamma$$

$N_i(x_i, y_i) = 1$ $N_i(x, y)$

$N_i(x_j, y_j) = 0$

节点   单元

$$u(x, y) = \sum_{i=1}^{n} N_i(x, y)u_i$$

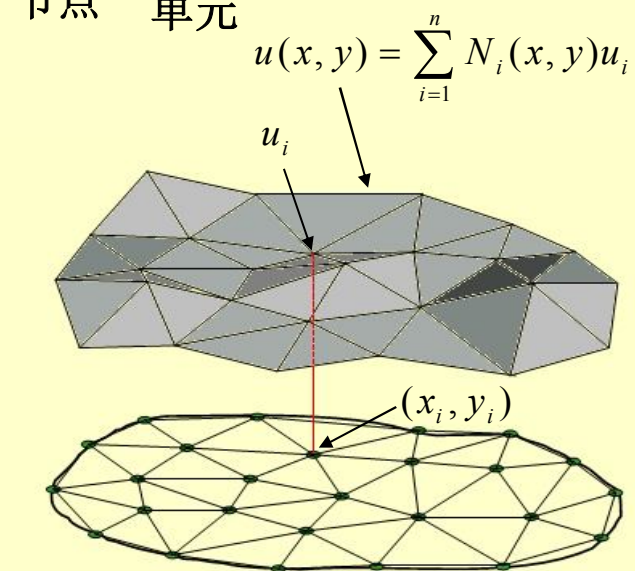u_i

(x_i, y_i)

将未知函数u近似为

$$u(x, y) = N_1 u_1 + N_2 u_2 + \cdots + N_n u_n$$

$$= \sum_{j=1}^{n} N_j(x, y)u_j$$

得到

$$\sum_{j=1}^{n} \int_{\Omega} \left(\frac{\partial v}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy \times u_j$$

$$= \int_{\Gamma_q} v\overline{q}d\Gamma$$

由于v是任意的，令v分别等于N_1到N_n，就得到n个代数方程

$$\sum_{j=1}^{n} \int_{\Omega} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy \times u_j = \int_{\Gamma_q} N_i \overline{q} d\Gamma \qquad (j = 1, 2, \cdots, n)$$
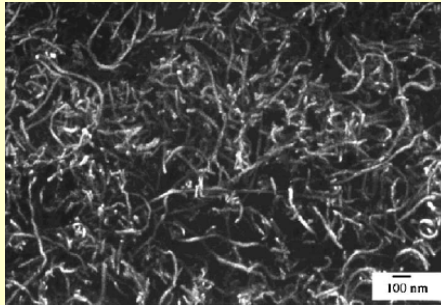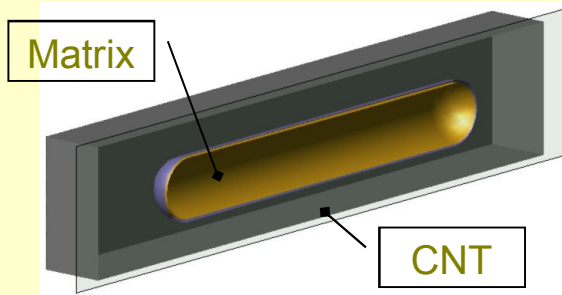
用矩阵表示为

$$A\mathbf{u} = \mathbf{f}$$

其中 $\quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad a_{ij} = \int_{\Omega} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dxdy$

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \qquad\qquad \mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{Bmatrix} \qquad f_i = \int_{\Gamma_q} N_i \overline{q} d\Gamma$$
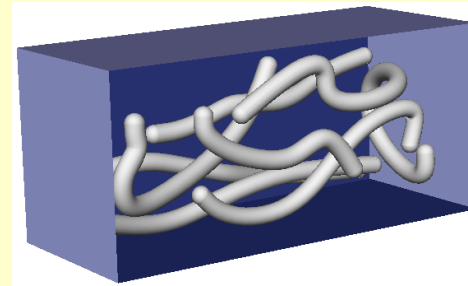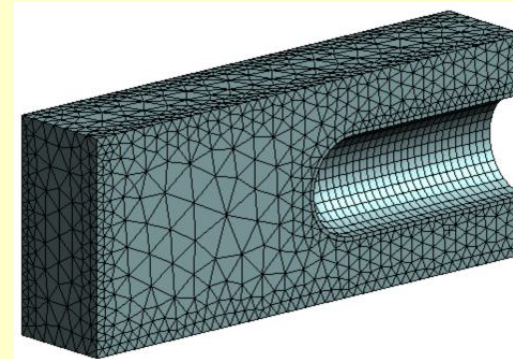
➢**Promising applications**

Nanotube-reinforced polymers

Matrix

CNT

Representative Volume Element (RVE)

➢**Numerical simulation model**

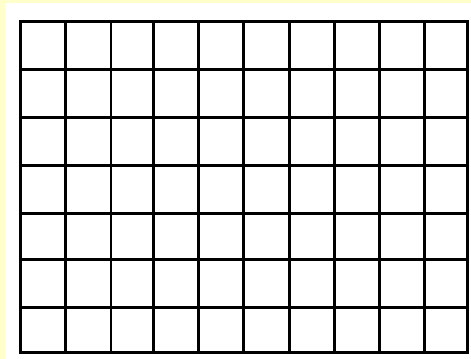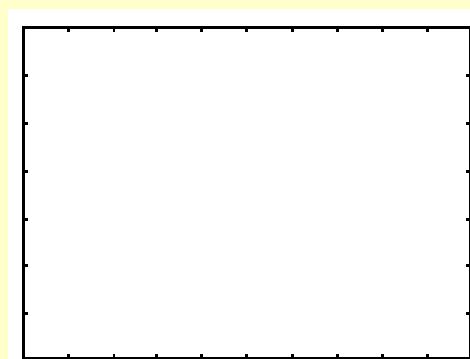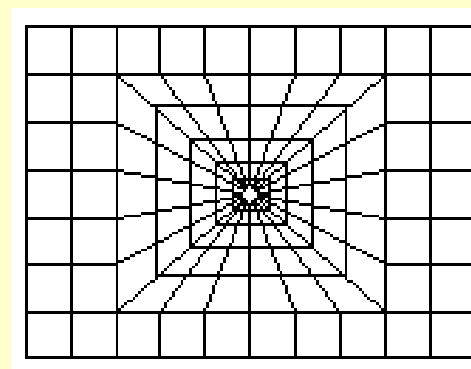**RVE including curved CNTs**

FEM/BEM Mesh for a quarter of an RVE

方板的有限元网格

方板的边界元网格

增加小孔后的有限元网格
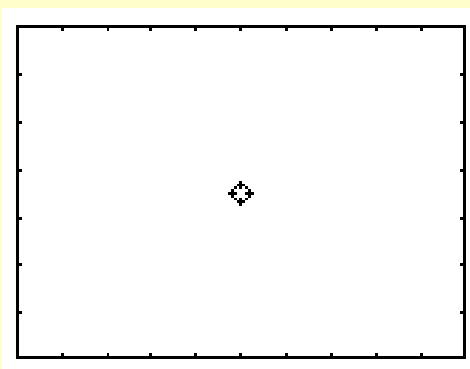
增加小孔后边界元网格
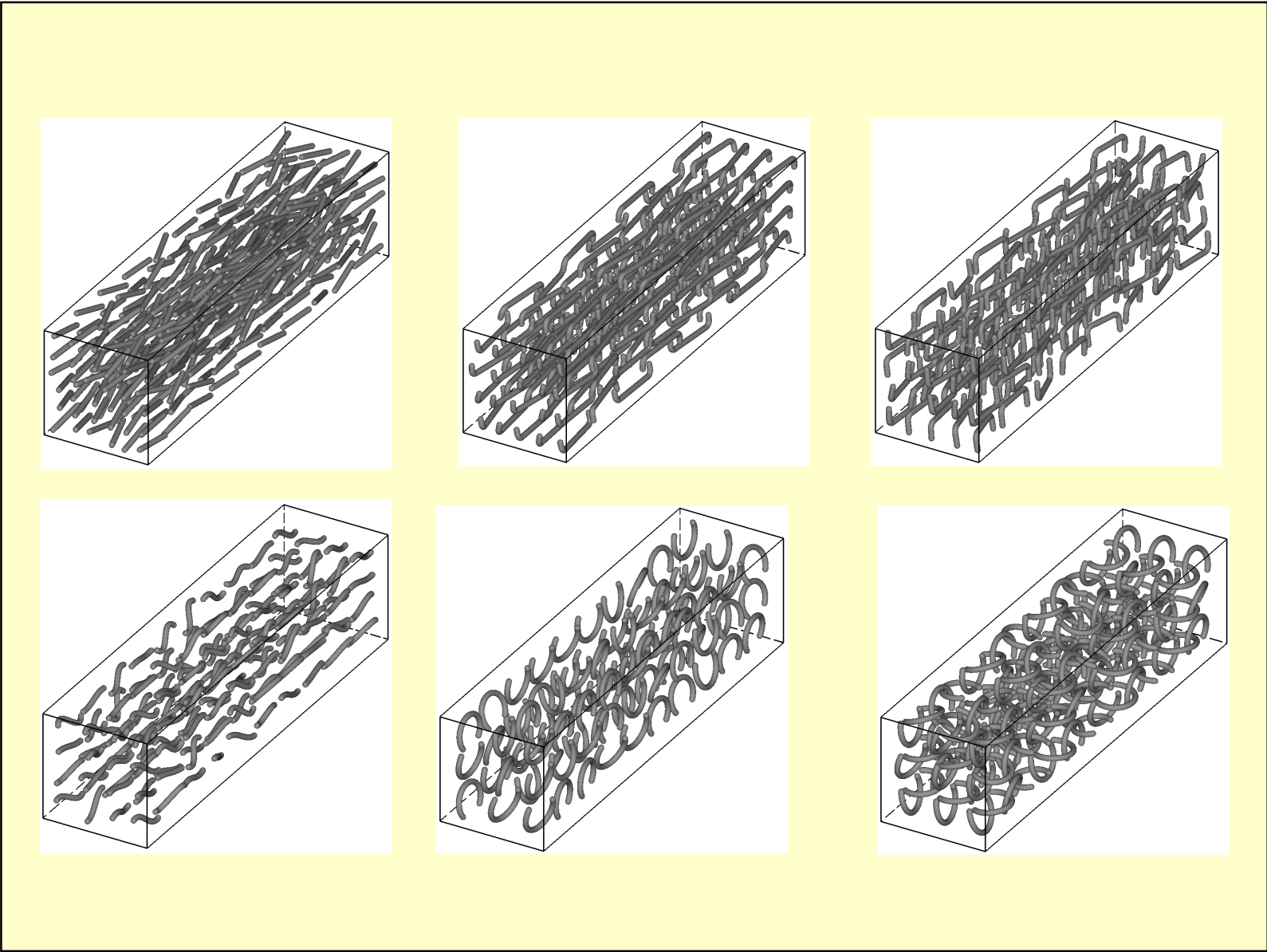
Delta Function
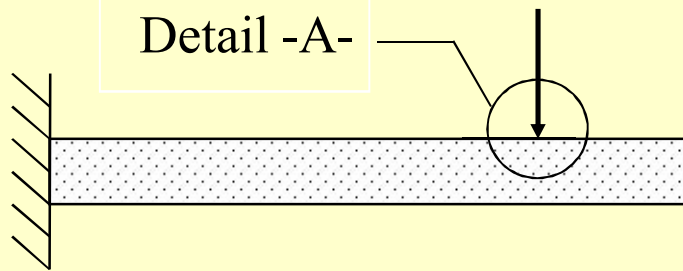
Detail -A-

Detail -A-
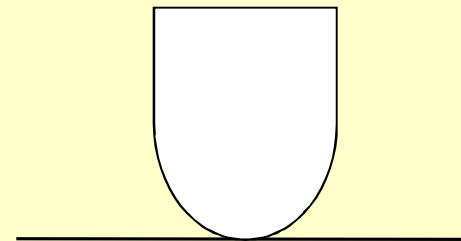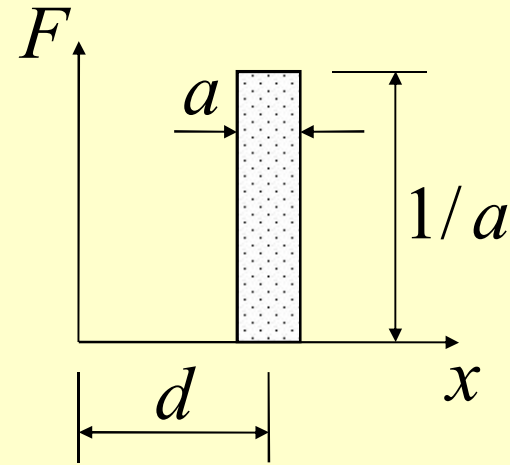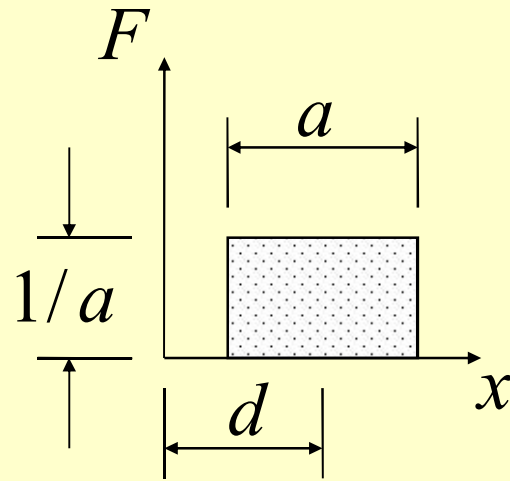
$$F(x,d,a) = \begin{cases} 0; & x < d - \dfrac{a}{2} \\[2em] \dfrac{1}{a}; & d - \dfrac{a}{2} \leq x \leq d + \dfrac{a}{2} \\[2em] 0; & x > d + \dfrac{a}{2} \end{cases}$$

$$\delta(x-d) = \lim_{\alpha \to 0} F(x,d,a)$$
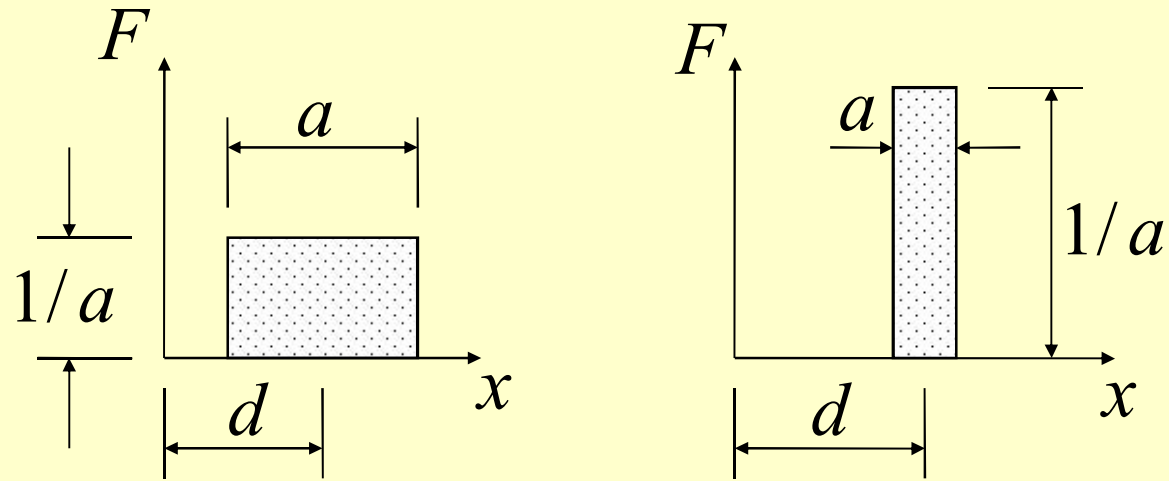
Selection property

$$I(d,a) = \int_{-\infty}^{+\infty} g(x)F(x,d,a)dx = \int_{d-a/2}^{d+a/2} g(x)F(x,d,a)dx$$

$$\int_{-\infty}^{+\infty} g(x)\delta(x-d)dx = \lim_{a \to 0} I(d,a) = \left\{ g(\alpha)\frac{1}{a} \right\} a = g(d)$$

边界元法（BEM）

二维稳态热传导问题
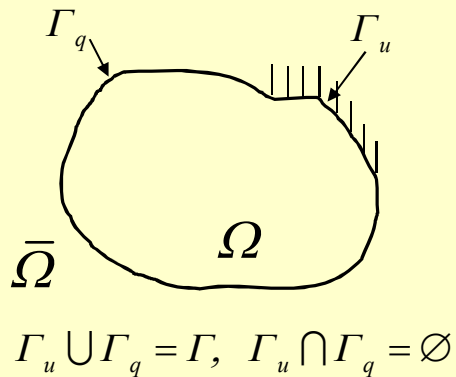
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad （在 \boldsymbol{\Omega} 内）$$

$$u - \bar{u} = 0 \qquad （在 \boldsymbol{\Gamma}_u 上）$$

$$\frac{\partial u}{\partial n} - \bar{q} = 0 \qquad （在 \boldsymbol{\Gamma}_q 上）$$

$$\Gamma_u \bigcup \Gamma_q = \Gamma, \ \ \Gamma_u \bigcap \Gamma_q = \varnothing$$

在有限元法的推导过程中，我们得到

$$\int_\Omega v \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) dxdy = -\int_\Omega \left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial u}{\partial y} \right) dxdy + \oiint_\Gamma v \frac{\partial u}{\partial n} d\Gamma$$

对 $\int_{\Omega}\left(\dfrac{\partial v}{\partial x}\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\dfrac{\partial u}{\partial y}\right)dxdy$ 继续用分部积分

$$\int_{\Omega}\dfrac{\partial v}{\partial x}\dfrac{\partial u}{\partial x}\,dxdy=\oiint_{\Gamma}\dfrac{\partial v}{\partial x}un_x d\Gamma-\int_{\Omega}\dfrac{\partial^2 v}{\partial x^2}udxdy$$

$$\int_{\Omega}\dfrac{\partial v}{\partial y}\dfrac{\partial u}{\partial y}\,dxdy=\oiint_{\Gamma}\dfrac{\partial v}{\partial y}un_y d\Gamma-\int_{\Omega}\dfrac{\partial^2 v}{\partial y^2}udxdy$$

从而有

$$\int_{\Omega}v\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right)dxdy=-\int_{\Omega}\left(\dfrac{\partial v}{\partial x}\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\dfrac{\partial u}{\partial y}\right)dxdy+\oiint_{\Gamma}v\dfrac{\partial u}{\partial n}d\Gamma$$

$$=\int_{\Omega}u\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}\right)dxdy-\oiint_{\Gamma}\left(\dfrac{\partial v}{\partial x}n_x+\dfrac{\partial v}{\partial y}n_y\right)ud\Gamma+\oiint_{\Gamma}v\dfrac{\partial u}{\partial n}d\Gamma$$

$$=\int_{\Omega}u\nabla^2 v d\Omega-\oiint_{\Gamma}\left(\dfrac{\partial v}{\partial n}u-v\dfrac{\partial u}{\partial n}\right)d\Gamma=0$$

即

$$\int_{\Omega} u\nabla^2 v d\Omega = \oiint_{\Gamma}\left(\frac{\partial v}{\partial n}u - v\frac{\partial u}{\partial n}\right)d\Gamma$$

Γ_q

Γ_u

P Q

Ω

$\bar{\Omega}$
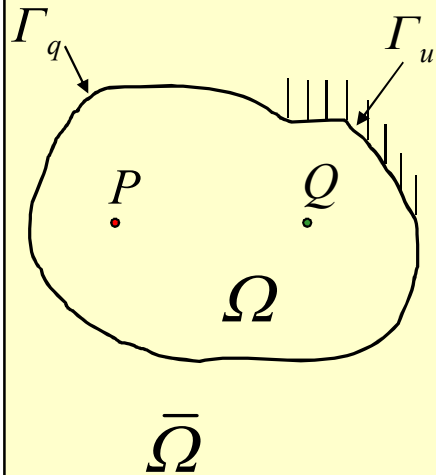
由于 v 是任意的，令

$$\nabla^2 v = \delta(Q,P) \qquad v = \frac{1}{2\pi}\ln r(Q,P)$$

$$r(Q,P) = \sqrt{\left(x(Q)-x(P)\right)^2 + \left(y(Q)-y(P)\right)^2}$$

由 δ 函数的性质

$$\int_{\Omega}\delta(Q,P)ud\Omega = \begin{cases} u(P), & \forall P \in \Omega \\ 0, & \forall P \in \bar{\Omega} \end{cases}$$

得到

$$u(P) = \oiint_\Gamma \left(\frac{\partial v(P,Q)}{\partial n} u(Q) - v(P,Q) \frac{\partial u}{\partial n}(Q) \right) d\Gamma(Q), \quad \forall P \in \Omega$$

$$0 = \oiint_\Gamma \left(\frac{\partial v(P,Q)}{\partial n} u(Q) - v(P,Q) \frac{\partial u}{\partial n}(Q) \right) d\Gamma(Q), \quad \forall P \in \bar{\Omega}$$
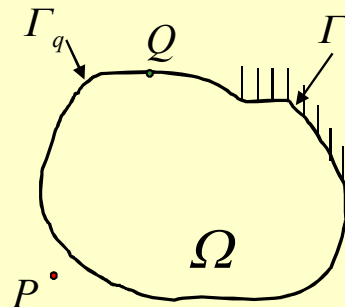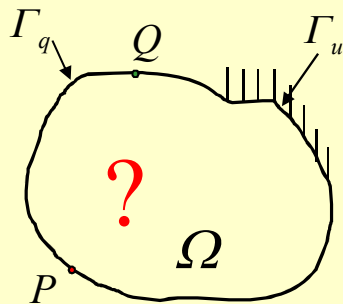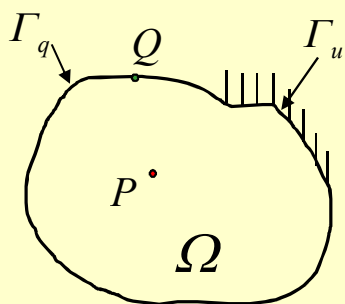
$$\int_\Omega \delta(Q,P) u\, d\Omega = \begin{cases} u(P), & \forall P \in \Omega \\ ?\,, & \forall P \in \Gamma \\ 0, & \forall P \in \bar{\Omega} \end{cases}$$

当P在边界上，如何得到边界积分方程？

有两种途径。

49

➢途径一：

$$\oint_{\Gamma - \Gamma_P + \Gamma_\varepsilon} \left(\frac{\partial v(P,Q)}{\partial n} u(Q) - v(P,Q) \frac{\partial u(Q)}{\partial n} \right) d\Gamma(Q) = 0$$

其中

$$\int_{\Gamma_\varepsilon} v(P,Q) \frac{\partial u}{\partial n}(Q) d\Gamma(Q) = \frac{1}{2\pi} \ln(\varepsilon) \varepsilon \int_{\theta_1}^{\theta_2} \frac{\partial u}{\partial n}(Q) d\theta$$

应用罗必达法则

$$\lim_{\varepsilon \to 0} \ln(\varepsilon) \varepsilon = \lim_{\varepsilon \to 0} \frac{\ln(\varepsilon)}{\varepsilon^{-1}} = \lim_{\varepsilon \to 0} \frac{\varepsilon^{-1}}{-\varepsilon^{-2}} = \lim_{\varepsilon \to 0} -\varepsilon = 0$$

又

$$\int_{\Gamma_\varepsilon} \frac{\partial v(P,Q)}{\partial n} u(Q) d\Gamma(Q) = \frac{1}{2\pi} \frac{-1}{\varepsilon} \int_{\theta_1}^{\theta_2} u(Q) \varepsilon d\theta$$

$$\lim_{\varepsilon \to 0} \frac{1}{2\pi} \frac{-1}{\varepsilon} \int_{\theta_1}^{\theta_2} u(Q) \varepsilon d\theta = \frac{\theta_1 - \theta_2}{2\pi} u(P)$$

及 $$\lim_{\varepsilon \to 0} (\Gamma - \Gamma_P) = \Gamma$$ 综合以上得到
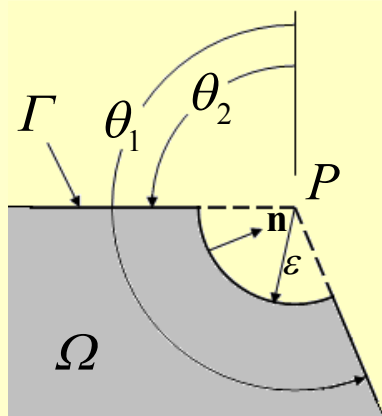
$$C(P)u(P) = \oint_{\Gamma} \left(\frac{\partial v(P,Q)}{\partial n} u(Q) - v(P,Q) \frac{\partial u(Q)}{\partial n} \right) d\Gamma(Q)$$

其中

$$C(P) = \begin{cases} 0, & \forall P \in \bar{\Omega} \\ 1, & \forall P \in \Omega \\ \dfrac{1}{2}, & P \text{ is on a smooth portion of } \Gamma \\ \dfrac{(\theta_2 - \theta_1)}{2\pi}, & P \text{ is at a sharp corner on } \Gamma \end{cases}$$
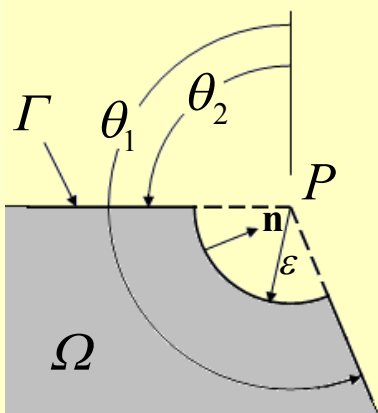
➢途径二：

$$u(P) = \oiint_\Gamma \left(\frac{\partial v(P,Q)}{\partial n} u(Q) - v(P,Q) \frac{\partial u}{\partial n}(Q) \right) d\Gamma(Q), \quad \forall P \in \Omega$$
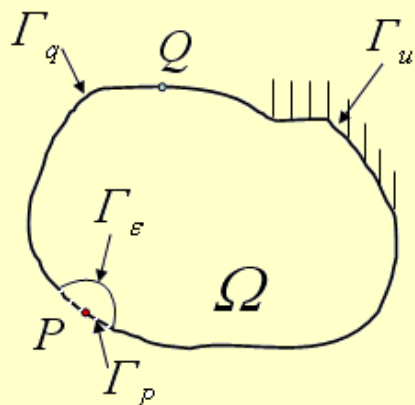
$$u(P) = \oiint_\Gamma \left(\frac{\partial v(P,Q)}{\partial n} \left[u(Q) - u(B) \right] - v(P,Q) \frac{\partial u}{\partial n}(Q) \right) d\Gamma(Q)$$

$$+ u(B) \oiint_\Gamma \frac{\partial v(P,Q)}{\partial n} d\Gamma(Q)$$

Γ_q　　Q　　Γ_u

P

B　　Ω

Γ

$d\Gamma$

$\cos\alpha\, d\Gamma$

$= \dfrac{\partial r}{\partial n} d\Gamma$

α　$\vec{n}$

$\vec{r}$

$d\Theta$

P
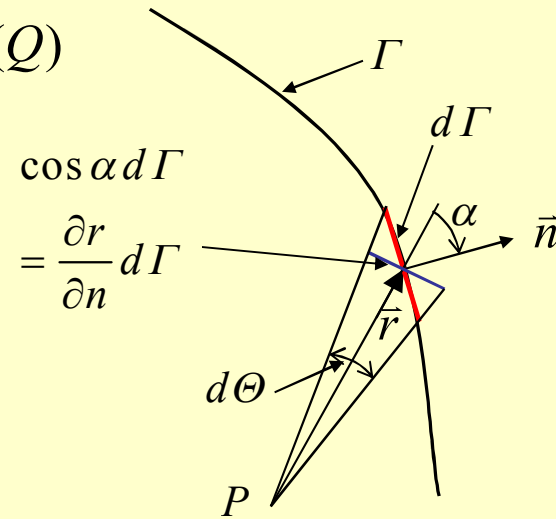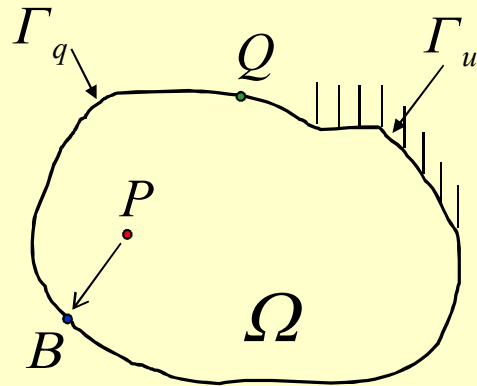
$$\oiint_\Gamma \frac{\partial v(P,Q)}{\partial n} d\Gamma(Q) = \oiint_\Gamma \frac{1}{2\pi} \frac{\partial \ln r}{\partial n} d\Gamma = \frac{1}{2\pi} \oiint_\Gamma \frac{1}{r} \frac{\partial r}{\partial n} d\Gamma = \frac{1}{2\pi} \oiint_\Gamma d\Theta = 1$$

于是得到

$$u(P) - u(B) = \oiint_\Gamma \left(\frac{\partial v(P,Q)}{\partial n} \big[u(Q) - u(B) \big] - v(P,Q) \frac{\partial u}{\partial n}(Q) \right) d\Gamma(Q)$$

让P趋近B，得到

$$0 = \oiint_\Gamma \left(\frac{\partial v(P,Q)}{\partial n} \big[u(Q) - u(P) \big] - v(P,Q) \frac{\partial u}{\partial n}(Q) \right) d\Gamma(Q)$$

$$N_i(t_i) = 1 \qquad N_i(t_j) = 0$$

$$t_1 \quad t_2 \qquad t_i \qquad t_j \qquad t_{n-1} \quad t_n \quad t_1$$

$$u_i \text{ or } q_i \qquad u(t) \approx \tilde{u}(t) = \sum_{i=1}^{n} N_i(t) u_i$$

$$t_1 \quad t_2 \qquad t_i \qquad t_j \qquad t_{n-1} \quad t_n \quad t_1$$

$$\Gamma_q \qquad \Gamma_j \qquad \Gamma_u$$

节点   单元

$$\Gamma = \sum_{k=1}^{N_e} \Gamma_k$$

将未知函数u近似为

$$u(t) = \sum_{j=1}^{n} N_j(t) u_j \qquad \frac{\partial u}{\partial n}(t) = q(t) = \sum_{j=1}^{n} N_j(t) q_j$$
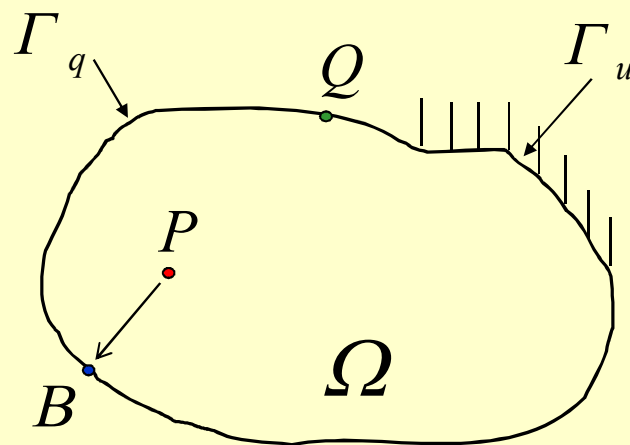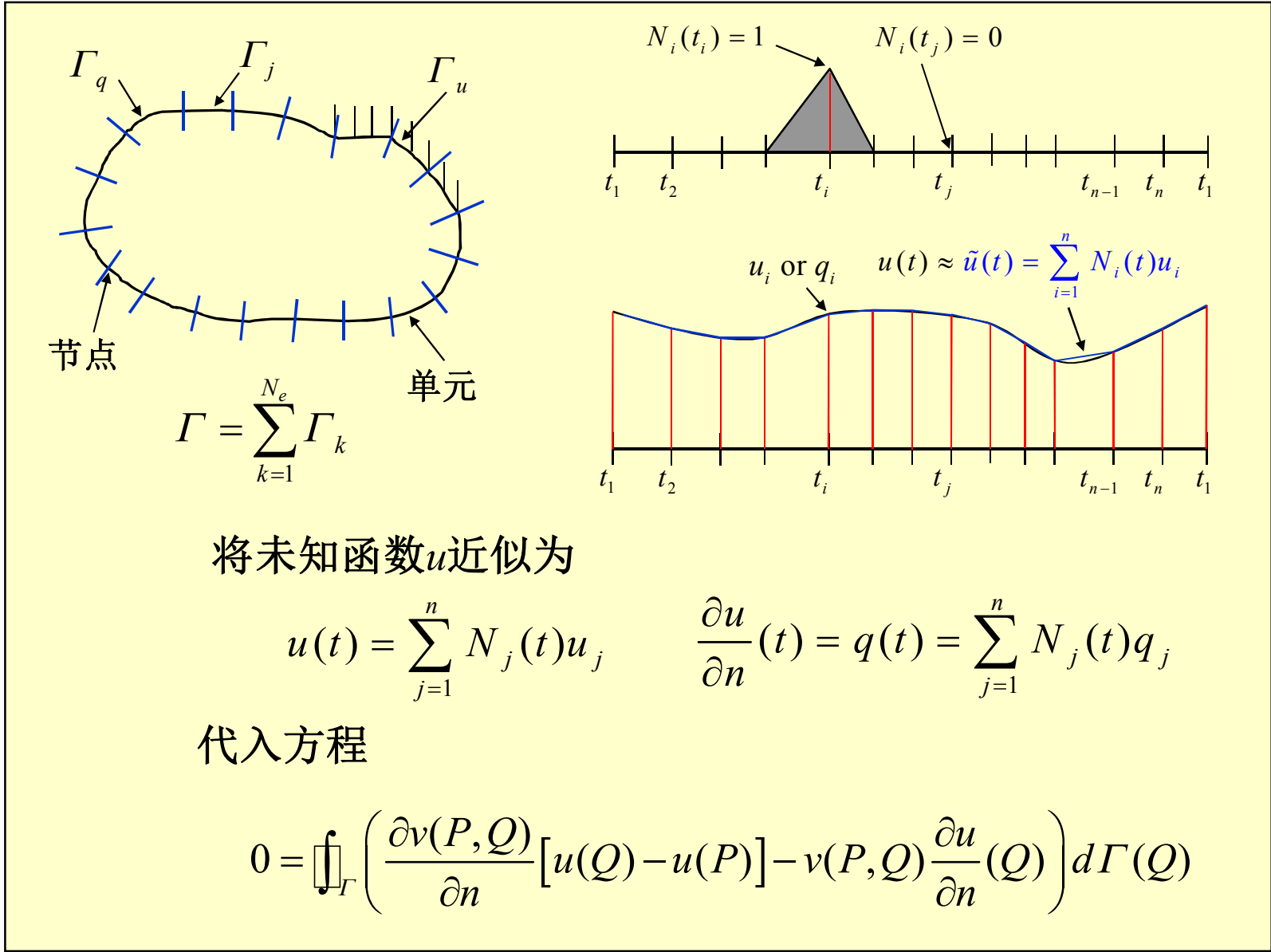
代入方程

$$0 = \oiint_{\Gamma} \left(\frac{\partial v(P,Q)}{\partial n} \left[u(Q) - u(P) \right] - v(P,Q) \frac{\partial u}{\partial n}(Q) \right) d\Gamma(Q)$$

得到

$$\sum_{k=1}^{N_e} \int_{\Gamma_k} \left(\frac{\partial \ln r(t,P)}{\partial n} \left[\sum_{j=1}^{n} N_j(t) u_j - u(P) \right] - \ln r(t,P) \sum_{j=1}^{n} N_j(t) q_j \right) d\Gamma(t) = 0$$

取P分别为所有节点t_i，$i=1,2,\dots,n$，得到代数方程组

$$(\mathbf{H} - \mathbf{D})\mathbf{u} - \mathbf{G}\mathbf{q} = \mathbf{0}$$

其中，$\mathbf{D}$是对角矩阵

$$H_{ij} = \sum_{k=1}^{N_e} \int_{\Gamma_k} \frac{\partial \ln r(t,t_i)}{\partial n} N_j(t) d\Gamma(t)$$

$$D_{ii} = \sum_{k=1}^{N_e} \int_{\Gamma_k} \frac{\partial \ln r(t,t_i)}{\partial n} d\Gamma(t)$$

$$\mathbf{u} = \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} \qquad \mathbf{q} = \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{Bmatrix}$$

$$G_{ij} = \sum_{k=1}^{N_e} \int_{\Gamma_k} \ln r(t,t_i) N_j(t) d\Gamma(t)$$

对任意节点，要么u_k已知，要么q_k已知。将已知量乘以矩阵相应的列，并移到方程的右边，得到

$$\mathbf{Ax = b}$$

其中，$\mathbf{x}$由未知的u_k和q_k组成。解出此方程就可以得到所有未知边界未知量。

再利用积分方程

$$u(P) = \sum_{k=1}^{N_e} \int_{\Gamma_k} \left(\frac{\partial \ln r(t,P)}{\partial n} \sum_{j=1}^{n} N_j(t) u_j - \ln r(t,P) \sum_{j=1}^{n} N_j(t) q_j \right) d\Gamma(t)$$

可求得域内任意点的函数值。

总结（1）） ——三种方法的比较

	FDM	FEM	BEM	Advances in BEM
方程推导	零次分部积分	一次分部积分	两次分部积分	
主要应用范围	流体	固体结构	固体结构	固体结构
网格离散	规则网格	不规则，适应性强	只在边界上离散，降维	
无限域问题（声场、地震等）	不适应	不适应	特别适应	
奇异性问题（裂纹分析）	不适应	不适应	特别适应	
非均质问题 非线性问题	适应	适应	不适应	DRM 适应
系数矩阵的计算和存储复杂度	稀疏矩阵 $O(n)$	对称稀疏矩阵 $O(n)$	非对称满阵 $O(n^2)$	FMM $O(n)$
方程求解计算量量级（迭代法）	$O(n)$	$O(n)$	$O(n^2)$	FMM $O(n)$
施加边界条件	不自然	比较自然	特别自然	
方法成熟度	已成熟	基本成熟	尚未成熟	热门研究中
计算精度	不高	高，但应力精度比位移精度低一阶	非常高，且应力和位移具有同样的精度	
与CAD接口	困难	困难	容易做到无缝连接	

总结（2）——数值求解的基本过程

➢ 导出微分方程等效的差分或积分形式（**FDM、FEM、BEM**）；

➢ 用网格将求解域离散；

➢ 用简单的函数（如多项式）逼近未知的连续场函数，其大小决定于网格节点值。这样就将一个无限自由度的连续场函数用有限个未知量表示，从而使得数值计算成为可能（可算化）；

➢ 通过数值微分（差分）和数值积分，将等效差分或积分方程转化为以节点值为未知数的线性代数方程组；

➢ 通过求解线性代数方程组得到所有节点值，从而得到未知场函数的近似解；

本课程的后续章节

- **函数近似**
  - 插值法
  - 函数拟合

- **数值积分和微分**

- **线性代数方程组数值求解**
  - 直接法
  - 迭代法